FAR BEYOND

MAT122

Integration by Substitution

Stony Brook University

Substitution Rule - Intro

Use **Substitution Rule** when there are 2 functions within the integration AND one is the derivative of the other.

ex. Find $\int 2x\sqrt{1+x^2}\,dx$

reorder $\int \sqrt{1+x^2}\,\underbrace{2x\,dx}$

$= \int \sqrt{u} \quad du$

ready for integration

$= \int u^{\frac{1}{2}}\,du$

$= \frac{2}{3}u^{\frac{3}{2}} + C$

$= \boxed{\frac{2(1+x^2)^{\frac{3}{2}}}{3} + C}$

let $u = 1+x^2$

$\frac{du}{dx} = 2x$ get differential $\Rightarrow du = 2x\,dx$

Do: check with differentiation

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + C$$

where $n \neq -1$

Substitution Rule

ex. Evaluate $\int \left(x^3+1\right)^5 \underbrace{3x^2 dx}$

$= \int u^5 du$

let $u = x^3 + 1 \Rightarrow du = 3x^2\, dx$

$= \dfrac{u^6}{6} + C$

$= \boxed{\dfrac{\left(x^3+1\right)^6}{6} + C}$

ex. Evaluate $\int \left(x^3+1\right)^5 \underbrace{x^2 dx}$

$= \int u^5 \cdot \dfrac{1}{3} du$

let $u = x^3 + 1 \Rightarrow du = 3x^2 dx$

$= \dfrac{1}{3}\int u^5 du$

$\Rightarrow \dfrac{1}{3} du = \underbrace{x^2 dx}$

$= \dfrac{1}{3} \cdot \dfrac{u^6}{6} + C$

$= \boxed{\dfrac{\left(x^3+1\right)^6}{18} + C}$

Substitution Rule – Example #2

ex. Evaluate $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$

separate $\displaystyle= \int \frac{1}{\sqrt{1-4x^2}} \cdot x\,dx$

$$\boxed{\text{let } u = 1-4x^2 \qquad du = -8x\,dx \qquad -\frac{1}{8}du = x\,dx}$$

$$= -\frac{1}{8}\int \frac{1}{\sqrt{u}}\,du$$

$$= -\frac{1}{8}\int u^{-\frac{1}{2}}\,du$$

$$= -\frac{1}{\cancel{8}} \cdot \cancelto{1/4}{2} \cdot u^{1/2} + C$$

$$= \boxed{-\frac{1}{4}\sqrt{1-4x^2} + C}$$

Substitution Rule – Example #3

ex. Calculate $\int e^{5x}\ dx$

$$\text{let } u = 5x \quad du = 5\ dx$$
$$\frac{1}{5}du = dx$$

$$= \frac{1}{5}\int e^{u}\ du$$

$$= \frac{1}{5}e^{u} + C$$

$$= \boxed{\frac{1}{5}e^{5x} + C}$$

Substitution Rule for Definite Integrals

ex. Evaluate $\int_0^4 \sqrt{2x+1}\ dx$

$$\boxed{\begin{aligned} &\text{let } u = 2x+1 \\ &\tfrac{1}{2}du = dx \end{aligned}}$$

change bounds from x to u

$x = 0: \quad u = 2(0)+1 = 1$

$x = 4: \quad u = 2(4)+1 = 9$

$$= \frac{1}{2}\int_1^9 u^{1/2}du$$

$$= \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}u^{3/2}\Big|_1^9 = \frac{1}{3}\left(9^{3/2} - 1^{3/2}\right)$$

$$= \frac{1}{3}(27-1) = \boxed{\frac{26}{3}}$$

or … keep bounds as x

$$= \frac{1}{2}\int_{x=0}^{x=4} u^{1/2}du$$

$$= \frac{1}{3}u^{3/2}\Big|_{x=0}^{x=4}$$

$$= \frac{1}{3}(2x+1)^{3/2}\Big|_{x=0}^{x=4}$$

$$= \frac{1}{3}\left(9^{3/2} - 1^{3/2}\right) = \boxed{\frac{26}{3}}$$

Substitution Rule for Definite Integrals – Example #2

ex. Evaluate $\int_1^2 \frac{dx}{(3-5x)^2}$ $= \int_1^2 \frac{1}{(3-5x)^2}\,dx = \int_1^2 (\underbrace{3-5x}_{u})^{-2}\,dx$

dx in numerator

let $u = 3-5x$

$du = -5\ dx$

$-\frac{1}{5}du = dx$

$$= -\frac{1}{5}\int u^{-2}\,du$$

$$= -\frac{1}{5}(-1)\,u^{-1}\Big|$$

substitute back in for x

$$= \frac{1}{5}\ \frac{1}{u}\Big| = \frac{1}{5}\left(\frac{1}{3-5x}\right)\Bigg|_1^2 = \frac{1}{5}\big[F(b)-F(a)\big]$$

$$= \frac{1}{5}\left[\frac{1}{3-5(2)} - \frac{1}{3-5(1)}\right]$$

$$= \frac{1}{5}\left[\frac{1}{-7} - \frac{1}{-2}\right]$$

$$= \frac{1}{5}\left[-\frac{1}{7}\cdot\frac{2}{2} + \frac{1}{2}\cdot\frac{7}{7}\right] = \frac{1}{5}\left[\frac{-2+7}{14}\right] = \frac{1}{\cancel{5}}\cdot\frac{\cancel{5}}{14} = \boxed{\frac{1}{14}}$$

Substitution Rule with lnx

ex. Evaluate $\int_1^e \frac{\ln x}{x}\,dx = \int_1^e \ln x \cdot \frac{1}{x}\,dx$

let $u = \ln x$

$du = \frac{1}{x}dx$

$= \int u\;du$

$u_a = \ln 1 = 0$

$u_b = \ln e = 1$

$= \int_0^1 u\;du$

$= \frac{u^2}{2}\Big|_0^1 = \frac{1}{2} - 0 = \boxed{\frac{1}{2}}$

ex. Evaluate $\int \frac{9}{8+9x}\,dx$ split up fraction

let $u = 8+9x$

$du = 9dx$

$= \int \frac{1}{8+9x} \cdot 9\;dx$

$= \int \frac{1}{u}\;du$ $\frac{1}{u}$ is derivative of $\ln u$

$= \ln|u| + C = \boxed{\ln|8+9x| + C}$

Review: Integration with u-Substitution

ex: $\int x \sqrt[3]{1+x^2}\,dx$

Do: re-order

$$= \int \sqrt[3]{1+x^2}\ x\ dx$$

$$= \int u^{1/3} \cdot \frac{1}{2}\ du$$

$$= \frac{1}{2}\int u^{1/3}\ du$$

$$= \frac{1}{2}\cdot\frac{3}{4}\cdot u^{4/3} + C$$

$$= \boxed{\frac{3}{8}\cdot\left(1+x^2\right)^{4/3} + C}$$

$$u = 1 + x^2$$

$$du = 2x\ dx$$

$$\frac{1}{2}du = x\ dx$$

Integration with *u*-Substitution – Practice #1

Do: $\int_{-1}^{2} \sqrt{x+2}\ dx$

$$\int_a^b f(x)dx = F(b) - F(a)$$

$$b^{\frac{x}{y}} = b^{\frac{1}{y}\cdot x} = \left(b^{\frac{1}{y}}\right)^x$$

i.e., evaluate root first

$$=\boxed{\frac{2}{3}}$$

Integration with *u*-Substitution – Practice #2

Do: Find the exact area under $f(x)=\dfrac{1}{x+1}$ between $x=0$ and $x=2$.

Hint:

$$\int_a^b \frac{1}{x}\,dx = \ln|x|$$

$$=\boxed{\ln 3}$$